MATHEMATICS

ON SYSTEMS OF MATING. IV

BY

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8. *A semi-random case.*

Here, the assumptions and notations will be as in section 7. Let j and k be fixed, $1 \leqslant j,\ k \leqslant n$. We shall say that the females of the types j and k are *indistinguishable* if, for females, the difference between the types j and k is due to a hidden factor (such as a recessive gene) which is not known and not noticeable in any way to any member of the population (not even to the females of the types j and k themselves), at least not before all couples have been formed already. However, the two numbers f_{ijr} and f_{ikr} may be quite different.

If indeed the females of types j and k are indistinguishable, then one should have

$$p_{ij}/p_j = p_{ik}/p_k, \qquad (i=1, \ldots, n), \tag{8.1}$$

for any mating matrix (p_{ij}). That is, the proportion p_{ij}/p_j, of females of type j which happen to mate with males of type i, should be equal to the corresponding proportion for females of type k, $(i=1, \ldots, n)$. Naturally, condition (8.1) is void when either $p_i=0$ or $p_k=0$. In a similar way,

$$p_{hj}/p_h = p_{ij}/p_i, \qquad (j=1, \ldots, n), \tag{8.2}$$

if males of the types h and i happen to be indistinguishable.

Having one or more pairs of indistinguishable types, that is, one or more conditions (8.1) or (8.2), the relevant problem would be to find the collection $\tilde{P}$ of all type distributions p such that there exists at least one nonnegative system $\{p_{ij}\}$ satisfying (7.3), (7.4) and, moreover, the conditions (8.1) or (8.2) on hand. Similarly, one would only be interested in systems of mating $\{G_{ij}(p)\}$ which satisfy the extra conditions

$$G_{ij}(p)/p_j = G_{ik}(p)/p_k, \qquad (i=1, \ldots, n), \tag{8.3}$$

or

$$G_{hj}(p)/p_h = G_{ij}(p)/p_i, \qquad (j=1, \ldots, n), \tag{8.4}$$

corresponding to (8.1) or (8.2), respectively. There is precisely one system of mating satisfying *all* conceivable conditions of the type (8.3) or (8.4), namely, the random mating system $G_{ij}(p)=p_ip_j$. It follows that the above collection $\tilde{P}$ is non-empty (at least when there are no non-trivial taboos, compare (2.22)).

In the present situation (unless in the sections 5–7), it would in general be an essential loss of generality to assume that $\{p_{ij}\}$ or $\{G_{ij}(\cdot)\}$ be symmetric (even when $f_{ijk}=f_{jik}$). For, consider a non-symmetric system $\{p_{ij}\}$ satisfying condition (8.1). Then the symmetrized system $\{p'_{ij}= =(p_{ij}+p_{ji})/2\}$ need not satisfy (8.1). More precisely, it satisfies (8.1) if and only if also $\{p''_{ij}=p_{ji}\}$ satisfies (8.1), that is, if and only if $\{p_{ij}\}$ treats also the male of the types j and k as being indistinguishable.

Let us now consider in some detail the special case $n=3$, $c=1$ (without non-trivial taboos) where for *both* males and females the pair (1, 2) is the *only* indistinguishable pair. For instance, this may be the case when the type of an individual is determined by its sex and further a single gene W or w with W dominant. For, if we let the genotypes WW, Ww, ww correspond to the types 1, 2 and 3, respectively, then within each sex the individuals of the types 1 and 2 will be of the same phenotype.

Presently, one is interested in the nonnegative 3×3 matrices (p_{ij}) satisfying

$$\sum_{i=1}^{3} p_{ij}=\sum_{i=1}^{3} p_{ji}=p_j, \qquad (j=1,2,3), \tag{8.5}$$

(say) and

$$p_{i1}/p_1=p_{i2}/p_2, \qquad p_{1j}/p_1=p_{2j}/p_2,$$

for all i, j, (at least when $p_1>0$, $p_2>0$). Introducing

$$a=p_1/(p_1+p_2), \qquad (0\leqslant a\leqslant 1), \tag{8.6}$$

it follows that (p_{ij}) must have the form

$$(p_{ij}) = \begin{pmatrix} a^2\varrho & a(1-a)\varrho & a\sigma \\ a(1-a)\varrho & (1-a)^2\varrho & (1-a)\sigma \\ a\sigma & (1-a)\sigma & \tau \end{pmatrix}; \tag{8.7}$$

(in particular, one automatically has $p_{ij}=p_{ji}$ in the present special case). In (8.7), a must satisfy $0\leqslant a\leqslant 1$ while ϱ, σ and τ are nonnegative numbers. From (8.5),

$$\varrho+\sigma=p_1+p_2, \qquad \sigma+\tau=p_3. \tag{8.8}$$

Hence,

$$\varrho+2\sigma+\tau=1, \tag{8.9}$$

at least when we normalize such that $\sum p_i=1$. By (8.6) and (8.8), if $p=(p_1, p_2, p_3)$ is given then the possible systems (p_{ij}) may be described by a single parameter σ, belonging to the interval

$$0\leqslant\sigma\leqslant \min\,(p_1+p_2, p_3). \tag{8.10}$$

The collection $\tilde{P}$ now consists of all the type distributions p such that there exists at least one nonnegative matrix of the form (8.7) satisfying (8.5), (8.9) and

$$(8.11)\qquad \sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij} f_{ijk} = \sum_{i=1}^{3} p_{ik},$$

for $k=1, 2, 3$. By $\sum_k f_{ijk}=1$, at least one of the equations (8.11) is redundant. Suppose for the moment that the system (8.11) is of rank 2. Then (8.7), (8.9) and (8.11) allow us to express ϱ, σ and τ, and, hence, the quantities

$$(8.12)\qquad p_1=a(\varrho+\sigma), \qquad p_2=(1-a)(\varrho+\sigma), \qquad p_3=\sigma+\tau$$

in terms of the single parameter a, $0\leqslant a\leqslant 1$. Moreover, a must be such that the corresponding numbers ϱ, σ and τ are nonnegative. The points (p_1, p_3) obtained in this way will usually form some curved arc in the (p_1, p_3)-plane. Hence, this arc represents the collection $\tilde{P}$ of all type distributions p which are stable under at least one matrix (p_{ij}) which treats the types 1 and 2 as being indistinguishable.

As an illustration, suppose that the numbers f_{ijk} are as in the very last example of section 6, see (6.20); in particular, $f_{ijk}=f_{jik}$. Then $\sum p_{ij}=1$ and (8.11) together are equivalent to

$$p_{11}+2p_{13}=\tfrac{3}{10}, \qquad p_{22}+2p_{12}=\tfrac{4}{10}, \qquad p_{33}+2p_{23}=\tfrac{3}{10}.$$

Using (8.7), one finds that $p\in\tilde{P}$ if and only if $p_1=a\varphi(a)$, $p_3=1-\varphi(a)$ for some number a satisfying

$$.4179=(\sqrt{129}-3)/20\leqslant a\leqslant\sqrt{3/7}=.6547.$$

Here,

$$\varphi(a)=(3+8a-7a^2)(20a-20a^3)^{-1}.$$

From now on, in this section, let us identify the types 1, 2, 3 with the genotypes WW, Ww and ww, respectively. Thus, we restrict ourselves to the classical case that $f_{ijk}=f_{jik}$ and that these numbers are as in (6.2). We assert that $p\in\tilde{P}$ if and only if

$$(8.13)\qquad \sqrt{\frac{p_3}{1-p_3}}\geqslant\frac{1-p_3-p_1}{1-p_3+p_1}=\frac{p_2}{2p_1+p_2}.$$

For, in the present case the system (8.11) is equivalent to the *single* equation $p_{22}=4p_{13}$. Consequently, a type distribution $p=(p_1, p_2, p_3)$ is in $\tilde{P}$ if and only if one can find *nonnegative* numbers ϱ, σ, τ and $0\leqslant a\leqslant 1$ satisfying (8.12) and

$$(1-a)^2\varrho=4a\sigma.$$

Necessarily,

$$p_3\geqslant\sigma=\left(\frac{1-a}{1+a}\right)^2(\varrho+\sigma)=\left(\frac{1-a}{1+a}\right)^2(1-p_3),$$

where $a = p_1/(p_1+p_2) = p_1/(1-p_3)$. This implies (8.13). Conversely, if (8.13) holds then

$$\varrho = \frac{4a}{(1+a)^2}(1-p_3) \quad , \quad \sigma = \frac{(1-a)^2}{(1+a)^2}(1-p_3)$$

and $\tau = p_3 - \sigma$ are all nonnegative. Obviously, these satisfy (8.12), showing that (8.13) is also sufficient.

The region in the (p_1, p_3)-plane defined by (8.13) turns out to be convex. It is bounded by the lines $p_1 + p_3 = 1$, $p_1 = 0$ and, further, the cubic curve, whose equation is obtained from (8.13) by taking there the equality sign. A few points (p_1, p_3) on this curve are (0, .500), (.0606), .400), (.228, .228), (.450, .100), (1.000, .000). The largest value p_2 is attained at the point (.1444, .3017), where $p_2 = .5539$. Hence, from our assumption that the types 1 and 2 are indistinguishable it follows that in an equilibrium situation there can be no more than 55 % heterozygotes.

REMARK. If all types were indistinguishable, so that only a purely random mating $p_{ij} = p_i p_j$ is allowed, there could be no more than 50 % heterozygotes. On the other hand, if there were no indistinguishable pairs at all then p_2 could be as high as 67 %, see section 6.

Still assuming the classical case, let us now discuss the behavior of the type distribution in the course of time when the population follows a partial system of mating $\{G_{ij}(\cdot)\}$ (compare section 7) which treats (within each sex) the types 1 and 2 as being indistinguishable. Such a system of mating associates to each type distribution p (belonging to a certain set Σ_3) a nonnegative matrix $(G_{ij}(p))$ of the special form (8.7), and having its row sums equal to p_i. Hence, (8.12) holds, thus, the quantities a, $\varrho+\sigma$ and $\sigma+\tau$ are already uniquely determined by the type distribution $p = (p_1, p_2, p_3)$. In specifying $(G_{ij}(p))$ it therefore would suffice to specify the number

$$u = p_{13}/p_1 = p_{23}/p_2 = \sigma/(\varrho+\sigma). \tag{8.14}$$

Observe that $p_3 = \sigma + \tau \geqslant \sigma$, while

$$\sigma = (\varrho+\sigma)\, u = (p_1+p_2)u = (1-p_3)u.$$

Hence, u must satisfy

$$0 \leqslant u \leqslant 1, \qquad u \leqslant p_3/(1-p_3), \tag{8.15}$$

and this is in fact the only restriction on u.

We conclude that the required partial system of mating $\{G_{ij}(\cdot)\}$ is *completely specified* by giving its domain Σ_3 and further a function $u = u(p)$ on Σ_3 satisfying (8.15). The only further restriction on Σ_3 and $u(\cdot)$ is that $p^{(t)} \in \Sigma_3$ must imply $p^{(t+1)} \in \Sigma_3$, where $p^{(t+1)}$ is obtained in the usual way from $p^{(t)}$ by applying the mating (p_{ij}) on hand, the latter being uniquely determined by $p^{(t)}$ and the number $u = u(p^{(t)})$.

More precisely, by (7.12),

$$p_3^{(t)} - p_1^{(t)} = d, \tag{8.16}$$

say, is independent of t. Moreover, with $u = u(p^{(t)})$,

$$p_3^{(t+1)} = d + \tfrac{1}{4}(1-u)(1-d)^2(1-p_3^{(t)})^{-1}. \tag{8.17}$$

In proving (8.17), put $p_i^{(t)} = p_i$ and $p_{ij}^{(t)} = p_{ij}$ for brevity. Then

$$p_3^{(t+1)} = \tfrac{1}{4}\, p_{22} + p_{23} + p_{33} = p_3 + \tfrac{1}{4}\, p_{22} - p_{13}.$$

Here, by (8.12) and (8.14), $p_{13} = up_1 = u(p_3 - d)$, while

$$p_{22} = (1-a)^2 \varrho = (1-a)^2(1-u)(\varrho + \sigma) = (1-a)^2(1-u)(1-p_3).$$

Finally,

$$a = p_1/(p_1 + p_2) = (p_3 - d)/(1 - p_3) = -1 + (1-d)/(1-p_3).$$

Combining these formulae, one obtains (8.17). Here, u will usually depend on $p^{(t)}$.

Let us now restrict ourselves to a partial system of mating such that

$$u(p) = u = \text{constant}, \qquad 0 < u < 1, \tag{8.18}$$

throughout Σ_3. By (8.14), this corresponds to a situation where, in all the subsequent generations, one and the same fraction of the individuals of type 1 mate with individuals of type 3.

We shall take Σ_3 as a set of the form

$$\Sigma_3 = \{(p_1, p_3) : p_3 - p_1 = d,\ \theta \leqslant p_3 \leqslant (1+d)/2\}, \tag{8.19}$$

where $-1 < d < 1$ and $d \leqslant \theta \leqslant (1+d)/2$ are given constants. They must be such that (8.15) holds for each point $p \in \Sigma_3$. Equivalently, θ must satisfy

$$\theta \geqslant u/(1+u). \tag{8.20}$$

Moreover, $p^{(t)} \in \Sigma_3$ must imply $p^{(t+1)} \in \Sigma_3$. From (8.17), we have $p_3^{(t+1)} \geqslant$ $\geqslant p_3^{(t)}$ precisely when

$$p_3^2 - (1+d)p_3 + d + \tfrac{1}{4}(1-u)(1-d)^2 \geqslant 0. \tag{8.20}$$

Here, $p_3 = p_3^{(t)}$ satisfies $0 \leqslant p_3 \leqslant (1+d)/2$. In this interval, the left-hand side of (8.20) has the unique zero

$$p_3^{(\infty)} = \frac{1+d}{2} - \frac{1-d}{2}\sqrt{u}. \tag{8.21}$$

In fact, it is now easily seen that

$$\begin{aligned} &p_3^{(t)} \leqslant p_3^{(t+1)} \leqslant p_3^{(\infty)} \text{ if } 0 \leqslant p_3^{(t)} \leqslant p_3^{(\infty)};\\ &p_3^{(\infty)} \leqslant p_3^{(t+1)} \leqslant p_3^{(t)} \text{ if } p_3^{(\infty)} \leqslant p_3^{(t)} \leqslant (1+d)/2. \end{aligned} \tag{8.22}$$

Consequently, as $t \to \infty$ we have that $p_3^{(t)}$ converges to $p_3^{(\infty)}$ in a *monotone* fashion. In particular, $p^{(t)} \in \Sigma_3$ implies $p^{(t+1)} \in \Sigma_3$ if and only if

$$\theta \leqslant p_3^{(\infty)} = \frac{1+d}{2} - \frac{1-d}{2}\sqrt{u}. \tag{8.23}$$

In order that there exist a value θ satisfying (8.20) and (8.23), we must have that

$$d = p_3 - p_1 \geqslant 1 - 2(1+u)^{-1}(1+\sqrt{u})^{-1}. \tag{8.24}$$

If this is the case, then the best value θ (making Σ_3 largest) would be $\theta = u(1+u)^{-1}$. We conclude that our particular system of mating (depending on the constant $0 < u < 1$) can be started and be continued indefinitely if and only if we start with a type distribution $p = p^{(0)}$ satisfying (8.24) and $p_3 \geqslant u(1+u)^{-1}$. The larger u, the smaller is the collection of admissible starting distributions.

In fact, in the (p_1, p_3)-plane this collection corresponds to a convex polyhedral region with the four extreme points

$$Q_1 = (0, 1) \quad , \quad Q_2 = \left(0, \frac{u}{1+u}\right),$$

$$Q_3 = \left(\frac{u}{1+u} - d, \frac{u}{1+u}\right) \quad , \quad Q_4 = \left(\frac{1-d}{2}, \frac{1+d}{2}\right);$$

here, $d = d(u)$ is given by (8.24) with equality sign. As u varies, the point Q_3 describes the cubic curve (8.13) (with equality sign). The line segment Q_1Q_3 corresponds to the collection of all type distributions which are stable under the system of mating defined by (8.18). As Q_3 varies, this line segment sweeps through the entire region $\tilde{P}$.

9. *Spinsters.*

Let us now study in some more detail the possibility that some individuals remain unmated. We shall restrict ourselves here to the special case $m = n = 3$, with $f_{ijk} = g_{ijk}$ and

$$\sum_{k=1}^{3} f_{ijk} = 1, \qquad (i, j = 1, 2, 3). \tag{9.1}$$

Consider a population, which is in equilibrium in the sense that it employs a mating scheme (p_{ij}) which leads to an offspring with the same type distribution as the population itself. We shall allow a fraction $1-c$ of the population to remain unmated, with $0 < c \leqslant 1$ as a *given* number. This would hardly differ from the case $c = 1$ if within each sex a fraction $1-c$ of *each* type would remain unmated. More interesting is the situation where, for some i and some j, an excessive number of males of type i and also an excessive number of females of type j remain unmated. This would normally only happen when there is some sort of obstruction for such males and females to mate with each other. For simplicity, we shall only consider an absolute obstruction, a so-called taboo, which dictates that $p_{ij}=0$. The set of all pairs (i, j) which are taboo is denoted by Γ. In the present section, as would be natural, we shall also require that

$$(i, j) \notin \Gamma \Rightarrow p_{i0}=0 \text{ or } p_{0j}=0. \tag{9.2}$$

Here, if there are N males and N females in the population, Np_{i0} stands for the number of males which are of type i and remain unmated; similarly, Np_{0j} for females. The number Np_k of males of type k is equal to the number of females of type k, (since $f_{ijk}=g_{ijk}$ and the population is in equilibrium). Thus,

$$p_k = \sum_{j=0}^{3} p_{kj} = \sum_{i=0}^{3} p_{ik}. \tag{9.3}$$

Moreover,

$$\sum_{j=1}^{3} p_{0j} = \sum_{i=1}^{3} p_{i0} = 1-c, \tag{9.4}$$

and

$$\sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} = c. \tag{9.5}$$

Finally, the equilibrium condition

$$\sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} f_{ijk} - cp_k = 0, \tag{9.6}$$

for $k=1, 2, 3$. Here, p_k is as in (9.3).

As to Γ, we will restrict ourselves to the following two cases.

(I). The only non-trivial pair in Γ is the pair (1, 3). Thus, males of type 1 are not allowed to mate with females of type 3, but females of type 1 are allowed to mate with males of type 3.

(II). The only non-trivial pairs in Γ are the two pairs (1, 3) and (3, 1).

The two cases can be treated simultaneously. In any case $p_{13}=0$. If $c=1$, then $p_{i0}=p_{0j}=0$ for all i and j, by (9.4). If $c<1$, then $p_{i0}>0$ for at least one i, $p_{0j}>0$ for at least one j, by (9.4). From (9.2), this can only happen for $i=1$, $j=3$ in case I, while in case II there is also the possibility that $i=3$ and $j=1$. However, one cannot have both $p_{10}>0$, $p_{03}>0$ and $p_{30}>0$, $p_{01}>0$, since the pairs (1, 1) and (3, 3) are never taboo; it suffices to consider the first situation (the second one being completely analogous).

Hence, in both case I and case II, we know the following: (i) $p_{13}=0$; (ii) $p_{20}=p_{30}=0$; $p_{01}=p_{02}=0$; (iii) $p_{10}=p_{03}=1-c$. In particular, (p_{ij}) is *never* symmetric.

Let us now formulate our problem in terms of the p_{ik} only. From (9.3), (9.4), we must have $L_k=0$ $(k=1, 2, 3)$, where

$$L_1=\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}f_{ij1}-c(p_{11}+p_{21}+p_{31}),$$

$$L_2=\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}f_{ij2}-c(p_{21}+p_{22}+p_{23}),$$

$$L_3=\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}f_{ij3}-c(p_{31}+p_{32}+p_{33}).$$

Moreover, from (9.3) with $k=2$, we have $L_4=0$ where

$$L_4=p_{21}+p_{23}-p_{12}-p_{32}.$$

Let Π_c^* denote the convex cone consisting of all non-zero nonnegative systems $\{p_{ij}; i, j=1, 2, 3\}$ satisfying the homogenous conditions $p_{13}=0$, $L_k=0$ $(k=1, 2, 3, 4)$. The required systems $\{p_{ij}\}$ all belong to Π_c^* and, moreover, satisfy (9.5). We claim that the converse is also true.

Let $\{p_{ij}\}$ be in Π_c^* and satisfy (9.5). Further, put $p_{10}=p_{03}=1-c$, $p_{i0}=0$, $p_{0j}=0$, otherwise. We must show that in the following diagram the k-th row sum p_k is equal to the k-th column sum p_k' $(k=0, 1, 2, 3)$:

$$\begin{bmatrix} 0 & 0 & 0 & 1-c \\ 1-c & p_{11} & p_{12} & 0 \\ 0 & p_{21} & p_{22} & p_{23} \\ 0 & p_{31} & p_{32} & p_{33} \end{bmatrix}. \tag{9.7}$$

This is obvious for $k=0$. For $k=2$, it follows from $L_4=0$. By $\sum p_k=\sum p_k'$, it remains to prove that $p_1=p_1'$, that is,

$$(1-c)+(p_{12}-p_{21}-p_{31})=0.$$

By (9.5), this is equivalent to $L=0$, where

$$L=(1-c)\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij}+c(p_{12}-p_{21}-p_{31}).$$

Indeed, using (9.1), we see that

$$L_1+L_2+L_3=L,$$

thus, our system $\{p_{ij}\}$ satisfies $L=0$.

It remains to determine the collection Π_c^* and its extreme rays. For, condition (9.5) can always be attained simply by replacing the p_{ij} by $p_{ij}'=\lambda p_{ij}$ with λ as a suitable positive constant.

As a byproduct, we see that necessarily

$$\tfrac{1}{2}\leqslant c\leqslant 1; \tag{9.8}$$

that is, no more that 50 % of the population can remain unmated.

For, if $c < \frac{1}{2}$ then in L each element p_{ij} would have its coefficient $\geqslant (1-c) - c > 0$, and $L=0$ would imply that $p_{ij}=0$ for all $i, j \geqslant 1$. It is easily seen that the case $c=\frac{1}{2}$ can arise only when $f_{311}=f_{333}=\frac{1}{2}$, $(f_{312}=0)$, namely, with $p_{31}=\frac{1}{2}$. It also follows from $L=0$ that

(9.9) $$\text{either } p_{21} > 0 \text{ or } p_{31} > 0,$$

provided $c < 1$.

From now on, in this section, *we shall restrict ourselves to the classical case* that the f_{ijk} are as in (6.2), $f_{ijk}=f_{jik}$. It will be convenient to interpret the types 1, 2, 3 in the usual way as genotypes WW, Ww and ww. If there are in total N males and N females, then the only individuals which do not get mated are $(1-c)N$ males of type WW and $(1-c)N$ females of type ww. Here, c is a given number satisfying $\frac{1}{2} \leqslant c \leqslant 1$.

Note that the number $2(1-c)N$ of W genes taken out of circulation is equal to the number of w genes taken out of circulation. Thus, one would expect that in an equilibrium situation

(9.10) $$p_1 = p_3 \text{ if } c < 1.$$

In fact, in the present case, the linear forms L_k reduce to

$$\begin{aligned}
L_1 &= p_{11} + \tfrac{1}{4}p_{22} + \tfrac{1}{2}p_{12} + \tfrac{1}{2}p_{21} - c(p_{11} + p_{21} + p_{31}),\\
L_2 &= \tfrac{1}{2}p_{12} + \tfrac{1}{2}p_{21} + \tfrac{1}{2}p_{22} + p_{31} + \tfrac{1}{2}p_{23} + \tfrac{1}{2}p_{32} - c(p_{21} + p_{22} + p_{23}),\\
L_3 &= \tfrac{1}{4}p_{22} + \tfrac{1}{2}p_{23} + \tfrac{1}{2}p_{32} + p_{33} - c(p_{31} + p_{32} + p_{33}),\\
L_4 &= p_{21} + p_{23} - p_{12} - p_{32}.
\end{aligned}$$

Let us also introduce

$$\begin{aligned}
L_5 &= p_{11} + p_{21} - p_{32} - p_{33},\\
L_6 &= p_{11} + p_{12} - p_{23} - p_{33}.
\end{aligned}$$

One has

(9.11) $$L_5 - L_4 = L_6$$

and

(9.12) $$(1-c)L_5 = L_1 - L_3 + \tfrac{1}{2}L_4.$$

A non-zero nonnegative system $\{p_{ij}\}$ is in Π_c^* if it satisfies $p_{13}=0$ and $L_k=0$ $(k=1, \ldots, 4)$. By (9.12), it satisfies $(1-c)L_5=0$ hence $L_5=0$ if $c<1$. This assertion in turn is equivalent to (9.10).

On the other hand, if $c=1$ then L_5 need not be zero at all while $L_3 = L_1 + \frac{1}{2}L_4$, from (9.12). Hence, in this case we have only the three conditions $L_1=0$, $L_2=0$, $L_4=0$. These in turn are equivalent to $p_{22}=2p_{31}$ together with $p_{13}=0$ and $\sum_j p_{ij} = \sum_j p_{ji}$. In other words, the quantities p_{11}, p_{21}, p_{31}, p_{32}, p_{33} can be arbitrary nonnegative numbers not all zero. Afterwards, the remaining p_{ij} are obtained from

$$p_{22} = 2p_{31}, \qquad p_{12} = p_{21} + p_{31}, \qquad p_{23} = p_{31} + p_{32}.$$

The normalized system is given by $p'_{ij}=p_{ij}/S$, where

$$S=\sum_{ij} p_{ij}=p_{11}+p_{33}+2(p_{21}+p_{32})+5p_{31}.$$

In case II one must choose $p_{31}=0$. The possible (p'_{ij}) form here a convex region with four extreme points, obtained by choosing all but one of the quantities p_{11}, p_{21}, p_{32}, p_{33} equal to 0. The stable distributions then correspond to the convex region in the (p_1, p_3)-plane determined by the 4 vertices $(1, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(0, 1)$. In case I there is the extra vertex $(\frac{1}{5}, \frac{1}{5})$ corresponding to the matrix

$$(p'_{ij}) = \begin{pmatrix} 0 & \frac{1}{5} & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & 0 \end{pmatrix},$$

obtained by taking $p_{31}>0$, $p_{11}=p_{21}=p_{32}=p_{33}=0$.

From now on, we assume that $\frac{1}{2}<c<1$ (if $c\leqslant\frac{1}{2}$ then Π_c^* is empty, compare (9.8)). By (9.12), we have $L_5=0$. Hence, by (9.11), we also have $L_6=0$. In fact, in the present case, Π_c^* may be regarded as the collection of all non-zero nonnegative systems $\{p_{ij}\}$ satisfying $p_{13}=0$, $L_1=0$, $L_2=0$, $L_4=0$, $L_5=0$. Next, we proceed to eliminate one of these. Namely,

$$L_2+\tfrac{1}{2}L_4=(1-c)\,(p_{21}+p_{23})+p_{31}-(c-\tfrac{1}{2})\,p_{22}.$$

Hence, we must have that

$$p_{22}=(c-\tfrac{1}{2})^{-1}\,[p_{31}+(1-c)\,(p_{21}+p_{23})], \tag{9.13}$$

(which corresponds to the result $p_{22}=2p_{31}$ when $c=1$). We may regard (9.13) as the *definition* of p_{22}, since $p_{22}\geqslant 0$ would be an automatic consequence of (9.13) and $p_{31}\geqslant 0$, $p_{21}\geqslant 0$, $p_{23}\geqslant 0$. Dropping the variable p_{22} we can simultaneously drop the condition $L_2=0$. In other words, Π_c^* may now be regarded as the collection of all 7-tuples

$$\{p_{11}, p_{33}, p_{23}, p_{32}, p_{31}, p_{12}, p_{21}\}$$

of nonnegative numbers (not all zero) which satisfy $L_1=0$, $L_4=0$, $L_5=0$. The quantity p_{22} occurring in L_1 is as in (9.13). Substituting (9.13) into L_1 one obtains the condition $L_1^*=0$, where

$$\begin{gathered} L_1^*=(4c-2)L_1+L_2+\tfrac{1}{2}L_4=2(1-c)\,(2c-1)\,p_{11}+ \\ +(1-c)\,p_{23}+(2c-1)\,p_{12}+(1+2c-4c^2)\,p_{31}-4c(c-\tfrac{3}{4})\,p_{21}. \end{gathered}$$

Note that

$$\begin{aligned} 1+2c-4c^2 &>0 \text{ if } 0<c<c_0, \\ &<0 \text{ if } c>c_0, \end{aligned} \tag{9.14}$$

where

$$c_0=(1+\sqrt{5})/4=.809>.750.$$

If $c<\frac{3}{4}$, then all the p_{ij} occurring in L_1^* (namely, p_{11}, p_{23}, p_{12}, p_{31}, p_{21}) would have a strictly positive coefficient. Thus, $L_1^*=0$ would imply that these p_{ij} are equal to zero. Afterwards, $L_5=0$ and (9.13) would imply that all the remaining p_{ij} are also equal to zero. Therefore, in order that Π_c^* be non-empty we must have that

$$c\geqslant\tfrac{3}{4}. \tag{9.15}$$

As is easily seen, the case $c=\frac{3}{4}$ is only possible with $p_{11}=p_{23}=p_{12}=p_{31}=0$ while $p_{21}=p_{32}=p_{22}$ $(=\frac{1}{4}$ if we normalize so that (9.5) holds). Thus, $c=\frac{3}{4}$ can be realized in both cases I and II. From now on, we assume that $c>\frac{3}{4}$. Then $L_1^*=0$ allows us to express p_{21} in terms of the remaining p_{ij}. Namely,

$$\begin{aligned}p_{21}=(4c^2-3c)^{-1}\{2(1-c)\,(2c-1)\,p_{11}+(1-c)\,p_{23}+(2c-1)\,p_{12}+\\ +(1+2c-4c^2)\,p_{31}\}.\end{aligned} \tag{9.16}$$

Before proceeding, let us first consider the important quantity $p_2=p_{21}+$ $+p_{22}+p_{23}$. Using (9.13), one has

$$p_2=(2c-1)^{-1}(p_{21}+p_{23}+2p_{31}). \tag{9.17}$$

Using (9.16), this yields

$$p_2=(4c^2-3c)^{-1}L_7, \tag{9.18}$$

where L_7 stands for the linear form

$$L_7=(2-2c)\,p_{11}+(2c-1)\,(p_{23}+p_{31})+p_{12}.$$

Next, by $p_3=p_1=p_{11}+p_{21}+p_{31}$, (9.16) and (9.17),

$$\sum_{i=1}^{3}\sum_{j=1}^{3}p_{ij}=(p_{11}+p_{12})+p_2+p_3=(4c-3)^{-1}L_8,$$

where

$$L_8=(4c-2)\,p_{11}+(p_{23}+p_{31})+(4c-1)\,p_{12}.$$

Therefore, if $\{p_{ij}\}$ is normalized so as to obtain (9.5) (equivalently, $p_1+p_2+p_3=1$), then

$$p_2=L_7/L_8. \tag{9.19}$$

Observe that, using $c>\frac{3}{4}$,

$$\frac{1-c}{2c-1}<\frac{1}{4c-1}<2c-1. \tag{9.20}$$

It follows from (9.19) that

$$(p_2)_{\min}=\frac{1-c}{2c-1}, \tag{9.21}$$

provided Π_c^* contains a system $\{p_{ij}\}$ satisfying $p_{23}=p_{31}=p_{12}=0$. In fact using the conditions $L_1^*=0$, $L_4=0$, $L_5=0$, we see that there is precisely one such system (in both cases I and II), namely, with

$$p_{11}=p_{33}=\lambda(4c^2-3c), \qquad p_{21}=p_{32}=\lambda(2-2c)\,(2c-1),$$

with λ as a positive constant. By (9.13), $p_{22}=4\lambda(1-c)^2$, hence, $p_2=2(1-c)\lambda$. In normalizing this solution, one takes $\lambda=(4c-2)^{-1}$.

In the further analysis, we shall distinguish between the following three cases:

(A) We are in case II, that is, $p_{31}=0$.

(B) We are in case I ($p_{31}>0$ allowed), while $\frac{3}{4}<c\leqslant c_0$.

(C) We are in case I, while $c_0<c<1$.

We now assert that, after normalization,

$$(9.22) \qquad \begin{aligned}(p_2)_{\max}&=2c-1 \text{ in case B,}\\ &<2c-1 \text{ in the cases A, C.}\end{aligned}$$

That $p_2\leqslant 2c-1$ is clear from (9.19), (9.20). Moreover, $p_2=2c-1$ can only happen when $p_{11}=p_{12}=0$. If so, then (by $L_6=0$, $p_{ij}\geqslant 0$) $p_{23}=p_{33}=0$ and (by $L_4=0$) $p_{21}=p_{32}$. Using $L_1^*=0$ and (9.14) we conclude: (i) In case A or C we have $p_{31}=p_{21}=0$. But then all the p_{ij} would be zero. Thus $p_2=2c-1$ is impossible in both cases A and C. (ii) In case B one must have $p_{32}=p_{21}=$ $=\lambda(1+2c-4c^2)$ and $p_{31}=\lambda(4c^2-3c)$, with λ as a positive constant. This is quite possible. In fact, after normalization, $\lambda=1$, $p_{22}=4c^2-2$, $p_1=p_3=$ $=1-c$, $p_2=2c-1$.

Later on we shall see that

$$(9.23) \qquad \begin{aligned}(p_2)_{\max}&=\tfrac{1}{2} \text{ in case A;}\\ &=(2c+1)/(4c+1) \text{ in case C,}\end{aligned}$$

(thus, $p_2>\frac{3}{5}$ is possible in case C). Let us first determine the convex cone Π_c^* in the cases A and B. In each of these cases, the coefficient a_{ij} of each variable p_{ij} on the right-hand side of (9.16) is nonnegative, compare (9.14); for later use, note that

$$a_{12}=(2c-1)(4c^2-3c)^{-1}>1.$$

We may regard (9.16) as the *definition* of p_{21}. In this way, one can simultaneously eliminate the variable p_{21} and the condition $L_1^*=0$.

Next, the condition $L_4=0$ is equivalent to

$$(9.24) \qquad p_{32}=(p_{21}-p_{12})+p_{23}.$$

By (9.16) and $a_{12}>1$, this expresses p_{32} as a linear form in p_{11}, p_{23}, p_{12}, p_{31} having nonnegative coefficients. In this way, one can eliminate the variable p_{32} and the condition $L_4=0$.

We conclude that, in the cases A and B, one may regard $\Pi_c{}^*$ as the collection of quintuples

$$\{p_{31},\ p_{11},\ p_{12},\ p_{23},\ p_{33}\}$$

of nonnegative numbers, not all zero, satisfying

$$p_{11}+p_{12}-p_{23}-p_{33}=0 \tag{9.25}$$

(and $p_{31}=0$ in case A). Each of the other p_{ij} is given as a linear form with nonnegative coefficients in terms of these five basic variables.

Afterwards, each $\{p_{ij}\}$ in Π_c^* may be renormalized so as to achieve (9.5). After such a normalization, p_2 is given by (9.19), showing that p_2 will attain its largest value at an extreme ray of Π_c^* (extreme point of Π_c).

From the above description of Π_c^*, see (9.25), it is clear that Π_c has five extreme points $\{p_{ij}^{(\nu)}\}$, $\nu=1, \ldots, 5$, in case B and only four extreme points $\{p_{ij}^{(\nu)}\}$, $\nu=1, \ldots, 4$ in case A. The extreme point $\{p_{ij}^{(5)}\}$ in case B (which is not present in case A) is obtained by choosing $p_{31}>0$, $p_{11}=p_{12}=$ $=p_{23}=p_{33}=0$. In fact, $\{p_{ij}^{(5)}\}$ is precisely the solution with $p_2=2c-1$, discussed in the paragraph following (9.22).

The four other extreme points are obtained by choosing $p_{31}=0$, one of p_{11}, p_{12} positive, one of p_{23}, p_{33} positive, in such a way that (9.25) holds. By (9.19) and (9.20), the one where p_2 is maximal has $p_{11}=0$ and p_{23}/p_{12} maximal. In particular, in case A the largest value of p_2 is obtained by choosing $p_{11}=p_{33}=0$, $p_{12}=p_{23}>0$ (besides $p_{31}=0$). In this case, $p_2=L_7/L_8=(2c)/(4c)=1/2$, proving the first part of (9.23).

It remains to consider case C, thus, assume $c_0<c<1$. The extreme points of Π_c with $p_{31}=0$ are the same as in case A (four in total), hence, it remains to determine the extreme points with $p_{31}>0$. The coefficients of L_1^*, L_4 and L_5 are as follows:

	p_{31}	p_{21}	p_{11}	p_{33}	p_{12}	p_{23}	p_{32}
L_1^*	$-\beta_{31}$	$-\beta_{21}$	$+\beta_{11}$	0	$+\beta_{12}$	$+\beta_{23}$	0
L_4	0	1	0	0	-1	1	-1
L_5	0	1	1	-1	0	0	-1

Here, the β_{ij} are strictly positive, namely,

$$\beta_{31}=4c^2-2c-1, \qquad \beta_{21}=4c^2-3c, \qquad \beta_{11}=(2-2c)(2c-1),$$

while $\beta_{12}=2c-1$, $\beta_{23}=1-c$. In the above scheme, let the h-th column be denoted as b_h $(h=0, 1, \ldots, 6)$. The extreme rays of Π_c^* correspond precisely to non-zero nonnegative solutions z of $z_0b_0+\ldots+z_6b_6=0$ such that the b_h with $z_h>0$ form an $3\times(m+1)$ matrix of rank m (see Lemma 3.2). Here, m may vary. In the present case only $m=2$ and $m=3$ are possible.

Presently, we are only interested in the extreme rays with $z_0=p_{31}>0$. If also $p_{21}>0$, then necessarily $m=3$,

$$p_{21}=p_{33}=p_{12},$$

p_{22} as in (9.13), $p_{ij}=0$ otherwise. In fact, there is precisely one such a solution, namely, with $p_{21}=\beta_{31}$ and $p_{31}=\beta_{12}-\beta_{21}=(1-c)(4c-1)>0$. By (9.19), the corresponding value p_2 is given by

$$p_2=(3-2c)/(4c-1).$$

Next, consider the extreme rays with $p_{31}>0$, $p_{21}=0$. If $p_{21}=0$, then (using $L_1^*=0$) one can eliminate p_{31} by expressing it as a linear form in p_{11}, p_{12}, p_{23} with positive coefficients. In this manner, one obtains three more extreme rays:

(ii) $p_{11}=p_{33}$, $p_{31}>0$, $p_{ij}=0$ otherwise;

(iii) $p_{12}=p_{23}$, $p_{31}>0$, $p_{ij}=0$ otherwise;

(iv) $p_{11}=p_{23}=p_{32}$, $p_{31}>0$, $p_{ij}=0$ otherwise;

(it is understood that p_{22} is to be computed as in (9.13), thus, $p_{22}>0$). For each of these, using (9.19), one can compute the value of p_2 after normalization. The largest value p_2 is obtained in case (iii), namely,

$$p_2=(2c+1)/(4c+1)>(3-2c)/(4c-1),$$

(since $c>c_0$). This proves the second part of (9.23).

Summarizing, Π_c has 4 extreme points in case A, 5 extreme points in case B, 8 extreme points in case C. Further, if $c<1$ then P_c is an interval defined by $p_1=p_3$, $\alpha\leqslant p_3\leqslant\beta$. Here, $\alpha=\frac{1}{2}[1-(p_2)_{\max}]$ and $\beta=\frac{1}{2}[1-(p_2)_{\min}]=(3c-2)/(4c-2)$ are known numbers depending on c.

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